CHAOS: ANOTHER TOOL FOR SYNTHESIS

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Abstract

In this paper we present a possible way to build a sound source based on a very simple map with chaotic properties. After analyzing the main mathematical aspects of the map, we show a method to interpret this system as an oscillator. Finally we mention the current problems that need a solution and we examine some of the possible applications.

1 Introduction

Real instruments' timbre usually exhibit a high level of complexity. Even a rough description of the spectral time evoluton (the envelopes of the main partials) might require a huge amount of data, but this is still far from a full description; for the timbre of a real sound always has some fluctuations, some level of uncertainty.

Sound designers must consider this if they want to create an artificial sound that has the same richness in sound quality as a real sound. On the other hand, playing an instrument doesn't require the continuous conscious manipulation of thousands of data, but normally a couple of simple gestures is just fine for playing music; the complexity of the sound depends broadly on the complexity of the links between the motion of the player and the parameters of the sound[1]. This is also something that must be considered when designing an instrument.

Since complex physical modelling requires so much computational power that this task can't usually be performed in real time, it would be essential to find algorythms that exhibit similar properties to acoustical instruments (stability of sound, small fluctuations, few control parameters), but don't require too much resources in the meantime. Chaotic sequences seem to be good candidates for this, since one can control the level of fluctuations in a smooth and efficient way simply by tuning the parameters of a chaotical system (like the Ljapunov-exponent).

In this paper we present such a chaotic map, the so-called Chirikov (or Standard) map [1].

[1] These links can usually be deduced using the laws of physics and the physical properties of the instruments

2 The Standard Map

The Chirikov-map is a $1 + 1$ dimensional classical dynamical system with one free parameter which describes a periodically 'kicked' pendulum in free field. The time evolution of the system is described by the dimensionless Hamiltonian

$$\mathcal{H}(p, x, t) = \frac{p^2}{2} + \delta_1(t) K \cos x \qquad (1)$$

where $\delta_1(t)$ is the 1–periodic Dirac-delta, which can be reformulated as a two dimensional compact discrete map:

$$\begin{aligned} p_{t+1} &= p_t + K \sin x_t \\ x_{t+1} &= p_t + x_t \end{aligned} \tag{2}$$

Analyzing (1) it turns out that it describes a kicked pendulum rotating in free field where the kicks happen at each time unit (according to the 1–periodicity of the Dirac-delta). This model is known as the Kicked Rotator [2], a physical model broadly used in theoretical solid-state physics. Since the phase of a pendulum is 2π–periodic, we can reformulate (2) taking the x coordinate modulo 2π, thus mapping the dynamics to a cylinder. Moreover, it can be shown that the dynamics remain the same even if we take p with modulo 2π, so that we map the system to a torus. However, for practical purposes instead of simply taking these values modulo 2π we will scale both x and p to the region $[-1;1]$. Our 'modified' standard map is therefore described by

$$\begin{aligned} p'_{t+1} &= \frac{1}{\pi}\left(\pi p'_t + K \sin(2\pi x'_t) \mod 2\pi\right) - 1 \\ x'_{t+1} &= \left(p'_t + x'_t \mod 2\right) - 1 \end{aligned} \tag{3}$$

A trajectory is defined by the initial conditions (p_0, x_0) and the 'kicking strength' parameter K. Different trajectories ranking to several values of K are plotted on Figures 1–6 (called 'Poincaré sections'). As we may see, there is a transition from regular motion, represented by closed trajectories like almost-straight lines or cycles, to chaotic, represented by the open, 'noisy' trajectories. For small values of K there are trajectories that split the phase space into disjoint mainfolds. It can be shown that the last of these 'splitting trajectories' – the so-called golden KAM[2] curve – disappears at $K_g \approx 0.971635$, which literally means that the map becomes fully chaotic above this value of K. This doesn't mean that above this value there is no regular motion in the phase space, however, the probability of finding a 'regular island' in the 'chaotic sea' drops exponentially with K above this critical value.

An important thing to mention is that this classical map can be quantized. This results in the Quantum-Kicked-Rotator (QKR) [2]. Without going into details we just mention here that numeric simulations need to take into consideration this quantized version of the map as the computer itself is a quantized device.

In the rest of this paper we will refer to the standard map according to (3), therefore we drop the apostrophes of x' and p'.

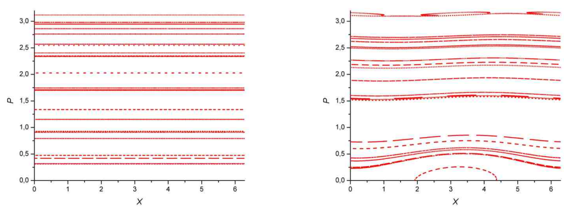

Figure 1: $K = 0$, $K = 0.05$

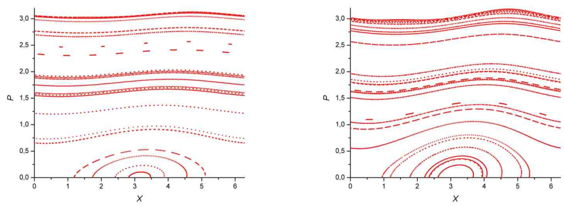

Figure 2: $K = 0.1$, $K = 0.2$

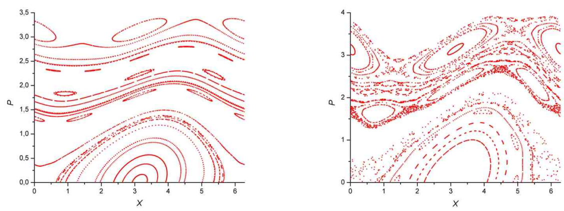

Figure 3: $K = 0.5$, $K = 0.97$

[2]Kolmogorov-Arnold-Moser

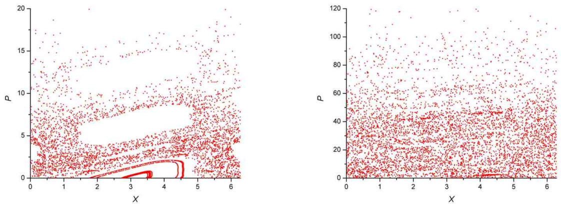

Figure 4: $K = 2$, $K = 5$

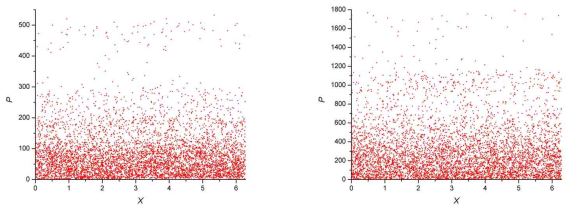

Figure 5: $K = 15$, $K = 50$

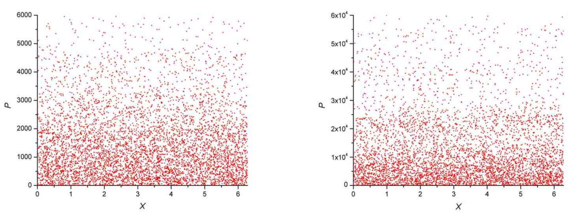

Figure 6: $K = 250, 2000$

3 Chaos as an Oscillator

One of the key benefits of the Standard Map is the periodicity of the coordinate parameter x. As it has been pointed out earlier, the phase space of the Standard Map corresponds to the motion of a periodically kicked pendulum. If the kicking strength K vanished, the pendulum would rotate freely, which corresponds to the uniform increasement of its phase x with a constant momentum (velocity) p.

Therefore, if we interpret the phase of the pendulum as the phase of a wavetable oscillator[3], for the $K = 0$ case we get an oscillator with a constant frequency. The p_0 coordinate of the initial point corresponds to the ω frequency of the oscillator according to the formula

$$p_0 = \frac{2f}{f_0}, \tag{4}$$

where f_0 is the sample rate of the system.

On the other hand, in the $K \to \infty$ limit when the whole phase space is chaotic, the oscillator behaves in a noisy way. Since the range of $K \sin(2\pi x_n)$ is $[-K; K]$, in the $K \to \infty$ limit the fractional part of this is practically a random number.

To summarize, the Standard Map serves as a phase generator with constant frequency in the $K = 0$ limit and as a noise generator in the $K \to \infty$ limit. Therefore we can use it as an interpolating oscillator between noisy and harmonic sounds.

Most interesting is the $K \approx K_g$ region. As we can see in Figure 3, in this region there are three kinds of trajectories:

- Trajectories that split the phase space. These correspond to sounds with a definite pitch with some small deviations in their frequencies.
- Trajectories that are restricted to a small area of the phase space (these can be seen as circles in the Poincaré sections). These trajectories correspond to sounds with strongly modulated frequencies.
- Trajectories in the 'chaotic sea'. These correspond to noise.

[3]In which case the discrete time steps of the Standard Map would correspond to the time quantization of the DSP.

Trajectories of the first two types are closed, the motion returns to the initial point in the phase space after a periodic time relevant to that trajectory, hence these trajectories are also called 'periodic orbits'. In contrary, trajectories of the last type are open, these trajectories fill the entire 'chaotic sea'. Since the computer running the DSP is quantized, it can happen that due to rounding errors a point traveling on a periodic orbit moves to an open trajectory and vice versa. This effect creates really nice, yet unpredictable sounds.

4 Conclusions

In this paper we presented a possible way to build a sound source based on a simple map with chaotic properties. This was the Chirikov (or Standard) Map which is related to the motion of the (Quantum) Kicked Rotator. We found that by tuning the 'kicking strength' parameter K of the model and the initial phase and momentum of the system we can create both harmonic and noisy sounds.

Although these results are very promising, there are many unanswered questions. First of all, although (4) provides a formula which connects the initial conditions of the system with the frequency of the generated sound for the $K = 0$ case, no formula is known for cases when $K \neq 0$. The other big obstacle is the lack of predictions for the trajectories, for it is practically impossible to tell whether a set of initial conditions (K, p_0, x_0) would result in a periodic orbit or an open one. It is even harder to guess whether an initial condition describing a periodic orbit corresponds to a trajectory that splits the entire phase space or if it corresponds to a small circle seen in the Poincaré sections. Finally, the biggest problem is that currently there's no way to describe the modifications of a particular trajectory – hence, the generated sound – when the value of K is changed, which is one of the most relevant musical questions. There is a strong suspicion that in theory it is not possible to solve this question.

In spite of these inconveniences, this chaotic oscillator might still be applied in electro-acoustic music. On one hand, it has powerful, yet unexplored possibilities as a low frequency oscillator, as it can generate almost-constant values with small and unpredictable deviations in the low range of the parameter K. On the other hand, for specific values of the initial conditions (K, p_0, x_0) it makes beautiful and unique noises which, with further processing, can serve as a fine basic material to compose with.

References

[1] Chirikov, B. & Shepelyansky, D. „Chirikov standard map", *Scholarpedia* **3(3):3550**, rev. #37486 (2008).

[2] Izrailev, F. M. „Simple models of quantum chaos: Spectrum and eigenfunctions", *Phys. Rep.* **196**, pp. 299–392 (1990).